Int. J. Solids Structures Vol. 24, No. 12, pp. 1235–1242, 1988
Printed in Great Britain.

0020–7683/88 $3.00+.00

NON-LINEAR FLEXURAL–FLEXURAL–TORSIONAL–EXTENSIONAL DYNAMICS OF BEAMS—II. RESPONSE ANALYSIS

M. R. M. CRESPO DA SILVA
Department of Mechanical Engineering, Aeronautical Engineering and Mechanics, Rensselaer Polytechnic Institute, Troy, NY 12180-3590, U.S.A.

(*Received* 13 *January* 1988; *in revised form* 20 *May* 1988)

Abstract—The importance of including the effects of non-linear inertia and curvature in the analysis of the motion of a beam with a constant distance between supports is assessed. For this, the non-linear response of clamped–clamped and clamped–pinned beams to a harmonic load of frequency near the undamped natural frequency of the beam is investigated by a perturbation method.

INTRODUCTION

The differential equations with cubic nonlinearities, developed in Part I of this work (Crespo da Silva, 1988) for an extensional beam with arbitrary boundary conditions, are specialized for a beam fixed (i.e. clamped or pinned) at one end and either fixed or arbitrarily supported at the other end. Here, for simplicity and for comparing results with other work presented in the literature, the material properties are taken to be constant along the beam. Also, the effects of the distributed mass moments of inertia are neglected. This implies that the torsional frequencies of the beam are much higher than its bending frequencies. The analysis presented here discloses the influence of higher order terms in the non-linear response of extensional beams. As an example, the planar response of an extensional beam to a periodic excitation, including the effect of all the cubic geometric nonlinearities, is analyzed in detail. The excitation can be either distributed along the beam, concentrated at several points along the span, or applied at its base (in which case, the base would also be moving).

SIMPLIFIED EQUATIONS OF MOTION FOR HIGH TORSIONAL FREQUENCIES

Consider a beam fixed at $x = 0$, i.e. $u(x = 0, t = t) = 0$, and either free or supported at $x = L$. The notation used here is the same as that in Part I (Crespo da Silva, 1988). Let the beam be subjected to a distributed force with components $Q_v(x, t)$ and $Q_w(x, t)$, along its two principal bending directions, and let damping in the system be viscous with virtual work given as $-c(\dot{v}\,\delta v+\dot{w}\,\delta w)$. For simplicity, the effect of the distributed mass moments of inertia are neglected. This implies that the beam's torsional natural frequencies are much higher than its bending natural frequencies. It is also assumed that the material properties of the beam are constant.

Let the "u boundary condition" at $x = L$ be expressed in the general form

$$K_u u(L, t)+LG_u(L, t)/EA = 0 \qquad (1)$$

where $K_u = \infty$ if $u(L, t) = 0$ (i.e. if the end at $x = L$ is fixed) and $K_u = 0$ if $u(L, t) \neq 0$ (i.e. if the end at $x = L$ is free to move with a non-zero u-displacement component).

With $Q_u(x, t) = 0$, the displacement $u(x, t)$ is given as

$$u(x, t) = xG_u(L, t)/EA-\frac{1}{2}\int_0^x (v'^2+w'^2)\,\mathrm{d}y+O(\varepsilon^3). \qquad (2)$$

This is obtained by integrating eqn (21) in Part I (Crespo da Silva, 1988), from $x = 0$ to x.

Evaluating eqn (2) at $x = L$ and combining the result with eqn (1), the following expression is obtained for $G_u(L, t)$:

$$LG_u(L, t)/EA = \frac{K_u}{2(1+K_u)} \int_0^L (v'^2 + w'^2)\, \mathrm{d}x. \tag{3}$$

By defining an angle of twist $\gamma(x, t)$ as

$$\gamma(x, t) = \theta_x(x, t) + \int_0^x v''w'\, \mathrm{d}y \tag{4}$$

and by considering the beam's end at $x = 0$ to be restrained against rotation, it follows that $\gamma(0, t) = 0$. To $O(\varepsilon^2)$, and with $Q_{\theta_x} = 0$, the following relationship is readily obtained for $\gamma(x, t)$ from eqn (18d) in Part I (Crespo da Silva, 1988). In eqn (5), $H_\gamma(x, t) = D_\zeta \gamma(x, t) + o(\varepsilon^2)$

$$D_\zeta \gamma(x, t) = xH_\gamma(L, t) + (D_\zeta - D_\eta) \int_0^x \int_L^y v''w''\, \mathrm{d}z\, \mathrm{d}y. \tag{5}$$

To obtain a general expression for $H_\gamma(L, t)$ let the "θ_x boundary condition" at $x = L$ be expressed as

$$K_\gamma \gamma(L, t) + LH_\gamma(L, t)/D_\zeta = 0 \tag{6}$$

where $K_\gamma = \infty$ if the end at $x = L$ is restrained against rotation, and $K_\gamma = 0$ if it is free to rotate. Evaluating eqn (5) at $x = L$ and combining the result with eqn (6), the following expression for $H_\gamma(L, t)$ is obtained:

$$LH_\gamma(L, t) = \frac{K_\gamma}{(1+K_\gamma)} (D_\eta - D_\zeta) \int_0^L \int_L^x v''w''\, \mathrm{d}z\, \mathrm{d}x. \tag{7}$$

By making use of eqns (2)–(5), and (7), the differential equations of motion, eqns (18b) and (18c) in Part I (Crespo da Silva, 1988) can be written as two integro-partial differential equations in the bending deflections $v(x, t)$ and $w(x, t)$. This was done analytically, by computer, with the aid of MACSYMA (Rand, 1984). To analyze the motion described by such equations, it is convenient to nondimensionalize the variables involved. With $(\ )^*$ denoting the non-dimensional form of $(\ )$, one lets

$$\begin{aligned} x^* &= x/L, \quad t^* = t[D_\eta/(mL^4)]^{1/2}, \quad c^* = cL^2/(mD_\eta)^{1/2} \\ \alpha^* &= \alpha/L, \quad Q_\alpha^* = Q_\alpha L^3/D_\eta \qquad (\alpha = v, w). \end{aligned} \tag{8}$$

With primes and dots denoting, from now on, partial differentiation with respect to x and t, respectively, and dropping the superscript * for convenience in notation, the normalized differential equations describing the flexural–flexural motion of an extensional beam with nonlinearities up to $O(\varepsilon^3)$ are then

$$\ddot{v} + c\dot{v} + \beta_y v'''' = Q_v(x, t) + (EAL^2/D_\eta) \frac{K_u v''}{2(1+K_u)} \int_0^1 (v'^2 + w'^2)\, \mathrm{d}x$$

$$+ \left\{ (1-\beta_y) \left[w'' \int_1^x v''w''\, \mathrm{d}y - w''' \int_0^x v''w'\, \mathrm{d}y - \frac{K_\gamma w''}{1+K_\gamma} \int_0^1 \int_1^x v''w''\, \mathrm{d}y\, \mathrm{d}x \right] \right.$$

$$-\frac{(1-\beta_y)^2}{\beta_\gamma}\left[w''\int_0^x\int_1^y v''w''\,dz\,dy-\frac{K_\gamma}{(1+K_\gamma)}(xw'')\int_0^1\int_1^x v''w''\,dy\,dx\right]'$$

$$-\beta_y v'(v'v''+w'w'')'+\beta_y\frac{K_u v'''}{1+K_u}\int_0^1(v'^2+w'^2)\,dx+\tfrac{1}{2}v'\int_1^x\left[\frac{yK_u}{1+K_u}\int_0^1(v'^2+w'^2)\,dx\right.$$

$$\left.-\int_0^y(v'^2+w'^2)\,dz\right]^{\cdot\cdot}dy\Bigg\}' \qquad (9a)$$

$$\ddot{w}+c\dot{w}+w'''' = Q_w(x,t)+(EAL^2/D_\eta)\frac{K_u w''}{2(1+K_u)}\int_0^1(v'^2+w'^2)\,dx$$

$$-\Bigg\{(1-\beta_y)\left[v''\int_1^x v''w''\,dy+v'''\int_0^x v''w'\,dy-v'''v'w'-\frac{K_\gamma v''}{1+K_\gamma}\int_0^1\int_1^x v''w''\,dy\,dx\right]$$

$$+\frac{(1-\beta_y)^2}{\beta_\gamma}\left[v''\int_0^x\int_1^y v''w''\,dz\,dy-\frac{K_\gamma(xv'')}{1+K_\gamma}\int_0^1\int_1^x v''w''\,dy\,dx\right]'+w'(v'v''+w'w'')'$$

$$-\frac{K_u w'''}{1+K_u}\int_0^1(v'^2+w'^2)\,dx-\tfrac{1}{2}w'\int_1^x\left[\frac{yK_u}{1+K_u}\int_0^1(v'^2+w'^2)\,dx-\int_0^y(v'^2+w'^2)\,dz\right]^{\cdot\cdot}dy\Bigg\}'$$

$$(9b)$$

where $\beta_y = D_\zeta/D_\eta$ and $\beta_\gamma = D_\xi/D_\eta$.

To $O(\varepsilon^2)$, eqns (9a) and (9b) reduce, for $K_u = \infty$, to eqns (5) and (6) in Ho, Scott and Eisley (1975) and to eqns (1) and (2) in Ho, Scott and Eisley (1976), where the forced and free non-linear non-planar motions of a pinned–pinned beam were investigated. For planar motion (i.e. either $v(x,t) = 0$ or $w(x,t) = 0$) these equations also reduce to eqns (20) in Nayfeh, Mook and Sridhar (1974) and to similar equations used by a number of investigators with $K_u = \infty$, to analyze the response of beams with a fixed distance between supports (Burgreen, 1951; Countryman and Kannan, 1985; Eisley, 1964; Eisley and Bennett, 1970; Ray and Bert, 1969; Srinivasan, 1966; Tseng and Dugundji, 1970; Woinovsky-Krieger, 1950).

The quantity $D_\eta/(EAL^2)$ that appears in the first bracketed term { } in eqns (9a) and (9b) is very small. It is the square of the radius of gyration (normalized by the length L of the undeformed beam) of the beam's cross section. For extensional beams, where $K_u \neq 0$, those terms are the dominant nonlinearities in the equations of motion. With $D_\eta/(EAL^2) = O(\varepsilon)$, they are $O(\varepsilon^2)$ while the remaining non-linear terms in the equations are $O(\varepsilon^3)$. These dominant nonlinearities, which are equal to $\lambda\alpha' \equiv \alpha' G_u(1,t)$ ($\alpha = v, w$), are due to midplane stretching of the beam. As indicated above, they are the same terms obtained by a number of other authors. For extensional beams, if one chooses $\varepsilon = D_\eta/(EAL^2)$ as in Nayfeh, Mook and Sridhar (1974), it is seen that the bending deflections are $O(\varepsilon)$ and, thus, very small. In the next section, the quantitative effect of both the $O(\varepsilon^2)$ and $O(\varepsilon^3)$ non-linear terms in the motion of an extensional beam is investigated in detail.

When $K_u = 0$ the beam behaves as inextensional to $O(\varepsilon^3)$. In this case, all nonlinearities for an initially straight beam are cubic. For $K_u = 0$ and $K_\gamma = 0$, eqns (9a) and (9b) reduce to eqns (5a) and (5b) in Crespo da Silva and Glynn (1978) for clamped–free beams.

RESPONSE CHARACTERISTICS OF A RESONANT MOTION

To assess the influence of the $O(\varepsilon^3)$ non-linear terms in the response of an extensional beam, consider the planar motion of pinned–pinned, pinned–clamped or clamped–clamped beams subjected to a distributed periodic excitation $Q_w(x,t) = Q(x)\cos(\Omega t+\tau)$. For $v(x,t) \equiv 0$, eqn (9b) reduces to

$$\ddot{w}+c\dot{w}+w'''' = Q(x)\cos(\Omega t+\tau)+\frac{K_u}{2(1+K_u)}(EAL^2/D_\eta)w''\int_0^1 w'^2\,dx-[(w'(w'w'')']'$$
$$+\left\{\frac{K_u}{1+K_u}w'''\int_0^1 w'^2\,dx+\tfrac{1}{2}w'\int_1^x\left[\frac{yK_u}{1+K_u}\int_0^1 w'^2\,dx-\int_0^y w'^2\,dz\right]^{\cdot\cdot}dy\right\}'. \quad (10)$$

To determine the steady-state response of the beam, and the character of the perturbed response about its steady state, use will be made of the perturbation method of multiple time scales (Nayfeh, 1981; Nayfeh and Mook, 1979). For this, three time scales $t_0 = t$, $t_1 = \varepsilon t$ and $t_2 = \varepsilon^2 t$ are introduced, where ε is a small arbitrary parameter used only for "bookkeeping purposes". With $K_u = \infty$, we let

$$EAL^2/(2D_\eta) \triangleq \beta_A/\varepsilon \quad (11a)$$

and expand the displacement $w(x, t)$ as

$$w(x, t_0, t_1, t_2; \varepsilon) = \varepsilon w_1(x, t_0, t_1, t_2)+\varepsilon^2 w_2(x, t_0, t_1, t_2)+\varepsilon^3 w_3(x, t_0, t_1, t_2)+\cdots \quad (11b)$$

To address the harmonic response of the beam including all the nonlinearities present in the differential equation, eqn (10), the damping coefficient and the excitation are expressed as

$$c = \varepsilon^2 c_2,\quad Q(x) = \varepsilon^3 Q_3(x). \quad (11c)$$

With $d(\ )/dt = (d_0+\varepsilon d_1+\varepsilon^2 d_2+\cdots)(\ )$, where $d_n(\ ) = \partial(\ )/\partial t_n$ $(n = 0, 1, 2, \ldots)$, eqn (10) then yields the following perturbation equations for $w_i(t_0, t_1, t_2)$.

$$O(\varepsilon):\quad d_0^2 w_1+w_1'''' = 0 \quad (12a)$$

$$O(\varepsilon^2):\quad d_0^2 w_2+w_2'''' = -2d_0 d_1 w_1+\beta_A w_1''\int_0^1 w_1'^2\,dx \quad (12b)$$

$$O(\varepsilon^3):\quad d_0^2 w_3+w_3'''' = Q_3(x)\cos(\Omega t_0+\tau)-2d_0 d_1 w_2-(d_1^2+2d_0 d_2)w_1-c_2 d_0 w_1$$
$$-[w_1'(w_1'w_1'')']'+\left\{w_1'''\int_0^1 w_1'^2\,dx+\tfrac{1}{2}w_1' d_0^2\int_1^x\left[y\int_0^1 w_1'^2\,dx-\int_0^y w_1'^2\,dz\right]dy\right\}'. \quad (12c)$$

Considering a one mode response, the solution to eqn (12a) is

$$w_1(x, t_0, t_1, t_2) = F(x)a(t_1, t_2)\cos[\omega t_0+\Phi(t_1, t_2)] \triangleq F(x)a\cos\Psi \quad (13a)$$

where $F(x)$ satisfies the differential equation $F''''-\omega^2 F = 0$. With

$$\int_0^1 F^2(x)\,dx = 1$$

for later convenience, $F(x)$ is given as:

pinned-pinned beam

$$F(x) = \sqrt{2}\sin(n\pi x);\quad n = 1, 2, \ldots; \quad (13b)$$

pinned-clamped or clamped-clamped beam

$$F(x) = \cosh(rx) - \cos(rx) - K[\sinh(rx) - \sin(rx)]$$
$$K = [\cosh(r) - \cos(r)]/[\sinh(r) - \sin(r)]. \tag{13c}$$

The frequency $\omega = r^2$ of the linearized oscillation satisfies the following relations:

$1 + (\cosh r)\cos r = 0$ for the clamped–clamped beam

$\tan r = \tanh r$ for the clamped–pinned beam.

To obtain an approximate solution for the non-linear response, it is convenient to introduce the quantities

$$w_{ii}(t_0, t_1, t_2) = \int_0^1 F(x) w_i(x, t_0, t_1, t_2)\,\mathrm{d}x \quad (i = 2, 3) \tag{14}$$

and then transform eqns (12b) and (12c) to ordinary differential equations by multiplying them by $F(x)$ and then integrating the result over the domain of x. With this procedure, eqn (12b) yields

$$d_0^2 w_{22} + \omega^2 w_{22} = 2\omega(d_1 a)\sin\Psi + a(2\omega d_1\Phi - 3\alpha\beta_A a^2/4)\cos\Psi - (\alpha\beta_A a^3/4)\cos(3\Psi) \tag{15a}$$

where

$$\alpha = -\int_0^1 FF''\,\mathrm{d}x \int_0^1 F'^2\,\mathrm{d}x = \left[\int_0^1 F'^2\,\mathrm{d}x\right]^2. \tag{15b}$$

The solvability conditions for eqn (15a), i.e. the necessary conditions for the solution to be periodic, are seen to be

$$d_1 a = 0 \tag{16a}$$
$$2\omega d_1\Phi = 3\alpha\beta_A a^2/4. \tag{16b}$$

To address the resonant case where Ω is near ω in eqn (12c), a detuning $\varepsilon^2\sigma_2 \ll 1$ is introduced as

$$\Omega = \omega(1 + \varepsilon^2\sigma_2). \tag{17}$$

With the transformation given by eqn (14), the $O(\varepsilon^3)$ eqn (12c) yields the following differential equation for w_{33}:

$$d_0^2 w_{33} + \omega^2 w_{33} = f_3\cos(\Psi + \gamma) - 2d_0 d_1 w_{22} - (d_1^2 + 2d_0 d_2)w_{11} - c_2 d_0 w_{11} - \alpha_2 w_{11}^3 - \alpha_1 w_{11}[w_{11} d_0^2 w_{11} + (d_0 w_{11})^2] \tag{18}$$

where $w_{11} = a(t_2)\cos\Psi$ and

$$\gamma = \omega\sigma_2 t_2 - \Phi + \tau \tag{19a}$$

$$f_3 = \int_0^1 F(x) Q_3(x)\,\mathrm{d}x \tag{19b}$$

$$\alpha_1 = -\int_0^1 F\left\{F' \int_1^x \left[y\int_0^1 F'^2\,\mathrm{d}x - \int_0^y F'^2\,\mathrm{d}x\right]\mathrm{d}y\right\}'\,\mathrm{d}x \tag{19c}$$

$$\alpha_2 = -\int_0^1 F\left[F'''\int_0^1 F'^2\,\mathrm{d}x - F'(F'F'')'\right]'\mathrm{d}x. \tag{19d}$$

By making use of eqns (16a) and (16b) and of the particular solution to eqn (15a) for w_{22}, the solvability conditions for eqn (18) are obtained as follows:

$$2\omega d_2 a + \omega c_2 a = f_3 \sin\gamma \tag{20a}$$

$$2\omega a d_2\Phi - [3\alpha\beta_A/(8\omega)]^2 a^5 + (\alpha_1\omega^2/2 - 3\alpha_2/4)a^3 = -f_3\cos\gamma. \tag{20b}$$

In terms of the original time t, the solvability conditions for eqn (10) are obtained by combining eqns (16a), (16b), (20a) and (20b). This yields the following differential equations for the amplitude and "phase" of the response:

$$2\omega\dot{a} \equiv 2\omega(\varepsilon d_1 a + \varepsilon^2 d_2 a + \cdots) = \varepsilon^2(f_3\sin\gamma - \omega c_2 a) \tag{21a}$$

$$2\omega\dot{\Phi} \equiv 2\omega(\varepsilon d_1\Phi + \varepsilon^2 d_2\Phi + \cdots) = 3\varepsilon\alpha\beta_A a^2/4 + \varepsilon^2\{[3\alpha\beta_A/(8\omega)]^2 a^5 - (\alpha_1\omega^2/2 - 3\alpha_2/4)a^3 - f_3\cos\gamma\}/a. \tag{21b}$$

Equations (21a) and (21b) are non-autonomous in the variables (a, Φ) but autonomous in the variables (a, γ) since $\dot{\Phi} = \omega\varepsilon^2\sigma_2 - \dot{\gamma}$. An equilibrium solution (a = constant = a_e, γ = constant = γ_e) to these equations correspond to an $O(\varepsilon)$ approximate solution to eqn (10). The amplitude–frequency relationship corresponding to this periodic solution is readily obtained from eqns (21a) and (21b) as

$$\{2\omega^2\varepsilon^2\sigma_2 + [\alpha_1\omega^2/2 - 3(\alpha_2 + \alpha\beta_A/\varepsilon)/4](\varepsilon a)^2 - [3\alpha(\beta_A/\varepsilon)/(8\omega)]^2(\varepsilon a)^4\}^2 + (\omega\varepsilon^2 c_2)^2 = [\varepsilon^3 f_3/(\varepsilon a)]^2. \tag{22}$$

The maximum amplitude, $\varepsilon a_{\max}$, of the response described by eqn (22) is then

$$\varepsilon a_{\max} = \varepsilon^3 f_3/(\omega\varepsilon^2 c_2) \tag{23a}$$

and it occurs when the detuning $\varepsilon^2\sigma_2 = \Omega/\omega - 1$ is equal to

$$\varepsilon^2\sigma_2 = \{[3\alpha(\beta_A/\varepsilon)/(8\omega)]^2(\varepsilon a_{\max})^2 - [\alpha_1\omega^2/2 - 3(\alpha_2 + \alpha\beta_A/\varepsilon)/4]\}(\varepsilon a_{\max})^2/(2\omega^2). \tag{23b}$$

An upper bound for the maximum amplitude of the motion can be readily obtained from eqn (23b). By imposing the condition $\varepsilon^2\sigma_2 \ll 1$, eqn (23b) yields as $\beta_A/\varepsilon \equiv EAL^2/(2D_\eta) \to \infty$

$$\varepsilon a_{\max} \ll 4\omega\left[\frac{\sqrt{2}}{3\alpha}D_\eta/(EAL^2)\right]^{1/2}. \tag{24}$$

Since $D_\eta/(EAL^2) \equiv I_\eta/(AL^2)$, where I_η is the principal area moment of inertia of the cross section about the η-axis, is a very small quantity, the upper bound for the maximum deflection for an extensional beam is of the order of the radius of gyration (normalized by L) of the beam's cross section. As a consequence of $\varepsilon a_{\max}$ being a very small quantity, and $\alpha\beta_A/\varepsilon$ being very large, the α_1, α_2 and $(\varepsilon a)^4$ terms in eqn (22) are negligible compared to the other terms in that equation. Thus, the nonlinearity due to midplane stretching is the

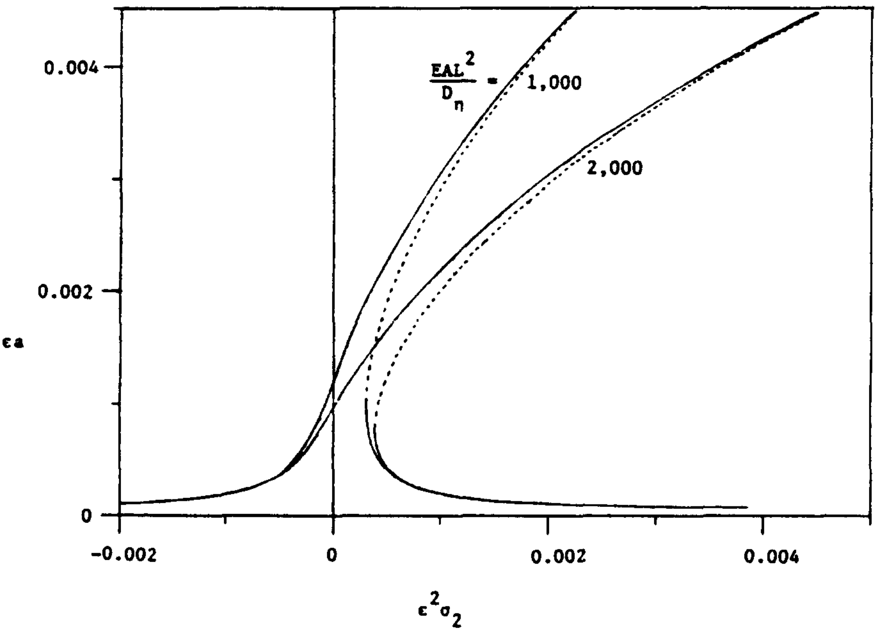

Fig. 1. Typical one-mode response of a beam with fixed ends ($\varepsilon^3 f_1 = 0.0002$, $c = 0.002$, $\omega = 22.37$, $\alpha = 151$, $\alpha_1 = 0.787$, $\alpha_2 = -465$): ———, stable; ·········, unstable.

dominant nonlinearity for extensional beams. Figure 1 shows the influence of EAL^2/D_n in a typical amplitude–frequency response for the three types of boundary conditions considered here. The values of ω, α, α_1 and α_2 for the first three modes are given in Table 1.

NUMERICAL INTEGRATION OF THE DIFFERENTIAL EQUATION OF MOTION

The analytical results obtained in the previous session can be verified by comparing them to an approximate numerical solution to eqn (10). For this, $w(x,t)$ is approximated as $w(x,t) = F(x)w_t(t)$. By applying Galerkin's procedure to eqn (10), the following ordinary differential equation is then obtained for $w_t(t)$, with $K_u = \infty$:

$$\ddot{w}_t(1+\alpha_1 w_t^2)+c\dot{w}_t+(\omega^2+\alpha_1\dot{w}_t^2)w_t+[\alpha_2+\alpha EAL^2/(2D_n)]w_t^3 = (\varepsilon^3 f_3)\cos(\Omega t+\tau). \quad (25)$$

Equation (25) immediately discloses that, in the limit as $EAL^2/D_n \to \infty$, the coefficient of the w_t^3 nonlinearity becomes $\alpha EAL^2/(2D_n)$. Since, as indicated by eqn (15b), α is not negative, this nonlinearity is always hardening for an extensional beam. Also, since the maximum value of w_t is much smaller than $\sqrt{(D_n/(EAL^2))}$, as disclosed by eqn (24) and by the values of ω and α shown in Table 1, the terms $\alpha_1 w_t^2$ and $\alpha_1[\mathrm{d}w_t/\mathrm{d}(\omega t)]^2$ are negligible compared to unity. Therefore, for an extensional beam, eqn (25) is essentially a damped Duffing's equation with a hardening nonlinearity.

Equation (25) was integrated numerically from $t = 0$ to T, where T was chosen to be large enough for the motion to reach a steady state. The numerical results are essentially indistinguishable from those obtained by the perturbation analysis of the previous section

Table 1. Natural frequencies and the Galerkin coefficients α, α_1 and α_2 for a uniform beam (first three modes)

	Clamped–clamped beam			Clamped–pinned beam			Pinned–pinned beam		
ω	22.37	61.67	120.90	15.42	49.96	104.24	π^2	$(2\pi)^2$	$(3\pi)^2$
α	151	2121	9782	132	1840	8843	π^4	$(2\pi)^4$	$(3\pi)^4$
α_1	0.789	38.2	140	7.32	45.5	146	0	0	0
α_2	−465	−5978	−32,140	−94	−1763	−11,000	0	0	0

and shown in Fig. 1. In the frequency range where the response is multivalued, the steady-state response exhibited by the beam depends on the initial conditions of the motion.

SUMMARY AND CONCLUSIONS

As shown previously (Crespo da Silva and Glynn, 1978), the dynamic response of inextensional beams can be significantly affected by non-linear inertia terms and by the non-linear contribution to the bending curvature in the differential equations of motion. These two effects can either reinforce or attenuate each other. Therefore, the question that naturally arises is what role these nonlinearities play in the motion of extension beams. To this author's knowledge, this question has been left unanswered in the literature to date. To address such a question, the non-linear response of an extensional beam subjected to a periodic excitation has been considered by taking into consideration the presence of these non-linear terms, and the nonlinearity due to midplane stretching. It has been shown that the effect of the latter nonlinearity is dominant and that neglecting the other non-linear terms in the differential equations of motion for an *extensional* beam introduce no significant error in the results of the analysis presented here. Also, it has been shown that, unlike the response of an inextensional beam, the single mode response of an extensional beam is always hardening.

REFERENCES

Burgreen, D. (1951). Free vibrations of a pin-ended column with constant distance between pin ends. *J. Appl. Mech., ASME Trans.* **18**, 135–139.

Countryman, M. and Kannan, R. (1985). Nonlinear forced vibrations of beams. *J. Appl. Mech., ASME Trans.* **52**, 163–166.

Crespo da Silva, M. R. M. (1988). Non-linear flexural flexural torsional extensional dynamics of beams—I. Formulation. *Int. J. Solids Structures* **24**, 1225–1234 (1988).

Crespo da Silva, M. R. M. and Glynn, C. C. (1978). Nonlinear flexural flexural torsional dynamics of inextensional beams. II: Forced motions. *J. Struct. Mech.* **6**, 449–461.

Eisley, J. G. (1964). Nonlinear vibrations of beams and rectangular plates. *ZAMP* **15**, 167–174.

Eisley, J. and Bennett, J. A. (1970). Stability of large amplitude forced motion of a simply supported beam. *Int. J. Non-linear Mech.* **5**, 645–657.

Ho, C. H., Scott, R. A. and Eisley, J. G. (1975). Non-planar, non-linear oscillations of a beam. I: Forced motions. *Int. J. Non-linear Mech.* **10**, 113–127.

Ho, C. H., Scott, R. A. and Eisley, J. G. (1976). Non-planar, non-linear oscillations of a beam. I: Free motions. *J. Sound Vibr.* **47**, 333–339.

Nayfeh, A. H. (1981). *Introduction to Perturbation Techniques*. Wiley-Interscience, New York.

Nayfeh, A. H. and Mook, D. T. (1979). *Nonlinear Oscillations*. Wiley-Interscience, New York.

Nayfeh, A. H., Mook, D. T. and Sridhar, S. (1974). Nonlinear analysis of the forced response of structural elements. *J. Acoust. Soc. Am.* **55**, 281–291.

Rand, R. H. (1984). *Computer Algebra in Applied Mathematics: an Introduction to MACSYMA*. Pitman, London.

Ray, J. D. and Bert, C. W. (1969). Nonlinear vibrations of a beam with pinned ends. *J. Engng Ind., ASME Trans.* **91**, 997–1004.

Srinivasan, A. V. (1966). Nonlinear vibrations of beams and plates. *Int. J. Non-linear Mech.* **1**, 179–191.

Tseng, W. Y. and Dugundji, J. (1970). Nonlinear vibrations of a beam under harmonic excitation. *J. Appl. Mech., ASME Trans.* **37**, 292–297.

Woinovsky-Krieger, S. (1950). The effect of an axial force on the vibration of hinged bars. *J. Appl. Mech., ASME Trans.* **17**, 35–36.